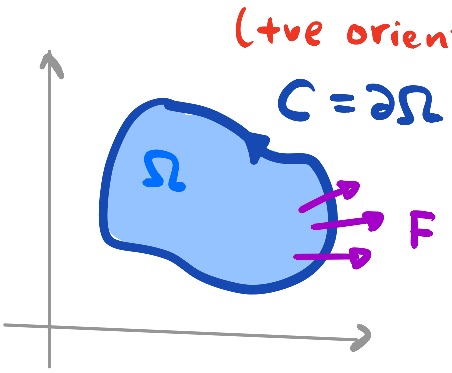
MATH 2028 Stokes and Divergence Theorem in $\mathbb{R}^3$

Goal: Statements of Stokes and Divergence Theorems and their applications.

Recall: Green's Theorem in $\mathbb{R}^2$



$$\int_C F \cdot d\vec{r} = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where $F = (P, Q)$.

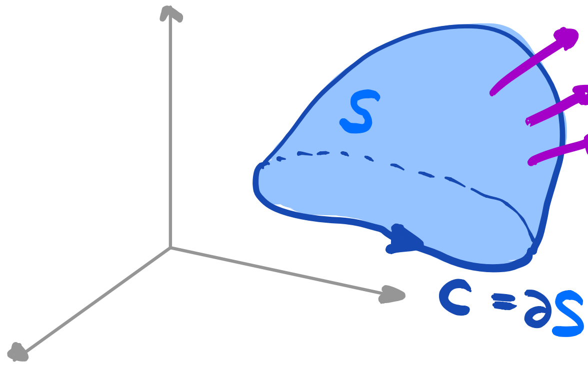
Q: Is there a similar theorem in $\mathbb{R}^3$?

A: Yes!

Stokes' Theorem

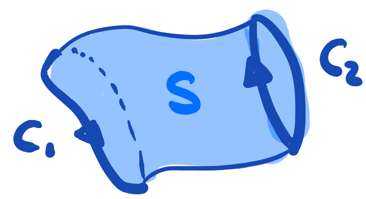
Let $S \subseteq \mathbb{R}^3$ be a surface with boundary $C = \partial S$ which is "positively" oriented (i.e. S lies on the left of C). Then, for any C^1 vector field F defined in an open set of $\mathbb{R}^3$ containing $S \cup C$, we have

$$\boxed{\int_C F \cdot d\vec{r} = \iint_S \operatorname{curl} F \cdot \vec{n} \, d\sigma}$$

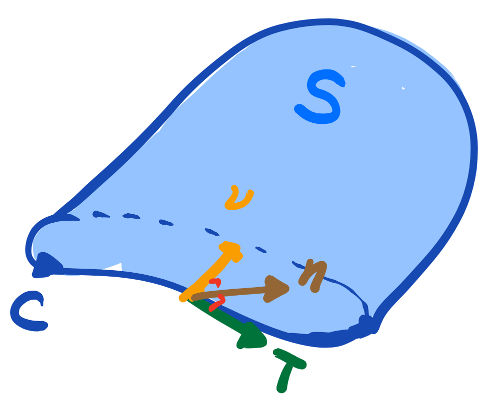


Remark: (1) ∂S may consist of more than one boundary curves.

E.g.) $\partial S = C_1 \cup C_2$



(2) The orientations of C and S are "compatible" in the sense that $\{T, \nu, n\}$ forms a "standard" orthonormal basis of $\mathbb{R}^3$ satisfying the "right-hand rule":



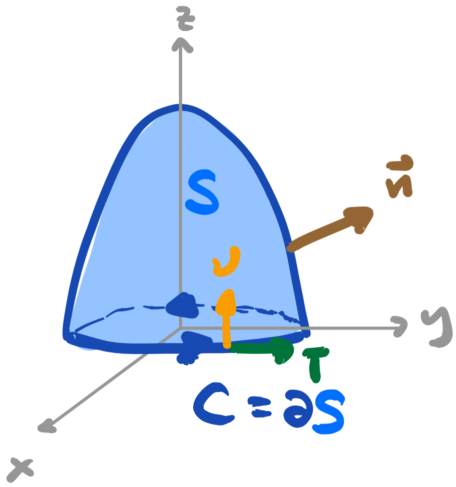
T : tangent to C

ν : tangent to S but normal to C and points into S

n : normal to S

<u>Example 1</u> : Compute $\iint_S \text{curl}\, F \cdot \vec{n}\, d\sigma$ for the vector field $F(x,y,z) = (z-y,\, x+z,\, -(x+y))$ and the surface S given by the paraboloid $z = 4 - x^2 - y^2$ with $0 \le z \le 4$.

<u>Solution:</u>



<u>Method 1</u> : Direct calculation.

Parametrize S by

$$g(u,v) = (u, v, 4-u^2-v^2)$$

where $u^2+v^2 \le 4$

$$\frac{\partial g}{\partial u} = (1, 0, -2u)$$

$$\frac{\partial g}{\partial v} = (0, 1, -2v)$$

$$\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = (2u, 2v, \underline{1})$$

points upward and outward!

(correct orientation!)

On the other hand,

$$\text{curl}\, F = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & x+z & -(x+y) \end{pmatrix} = (-2, 2, 2)$$

Therefore,

$$\iint_S \operatorname{curl} F \cdot \vec{n}\, d\sigma = \iint_{\{u^2+v^2\leq 4\}} (-2,2,2)\cdot(2u,2v,1)\, du\, dv$$

$$= \iint_{\{u^2+v^2\leq 4\}} (-4u+4v+2)\, du\, dv$$

$$= \int_0^{2\pi}\int_0^2 (-4r\cos u + 4r\sin u + 2)\cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left(-\frac{32}{3}\cos u + \frac{32}{3}\sin u + 4\right) d\theta$$

$$= 8\pi \quad \#$$

Method 2 : Apply Stokes Theorem.

$$\iint_S \operatorname{curl} F \cdot \vec{n}\, d\sigma = \int_C F \cdot d\vec{r}$$

Parametrize C (with correct orientation!)

by $\gamma(t) = (2\cos t, 2\sin t, 0)$

where $0 \leq t \leq 2\pi$

Then. $\gamma'(t) = (-2\sin t,\ 2\cos t,\ 0)$

$$F \circ \gamma(t) = (-2\sin t,\ 2\cos t,\ -2\cos t - 2\sin t)$$

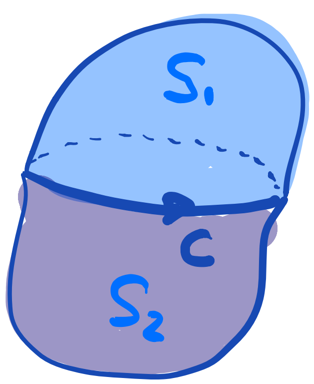
$$\Rightarrow \int_C F \cdot d\vec{r} = \int_0^{2\pi} F\circ\gamma(t) \cdot \gamma'(t)\, dt$$

$$= \int_0^{2\pi} 4\sin^2 t + 4\cos^2 t \, dt$$

$$= 8\pi \quad * \qquad \text{(SAME answer!)}$$

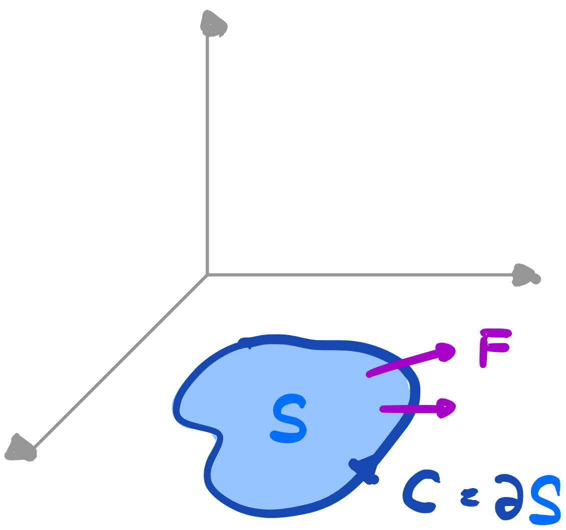
Suppose two surfaces S_1, S_2 in $\mathbb{R}^3$ are bounded by the same curve C and lie on "opposite" sides of C. THEN: by Stokes Theorem:

$$\int_{\partial S_1} F\cdot d\vec{r} = \iint_{S_1} \text{curl}\, F \cdot \vec{n}\, d\sigma = -\iint_{S_2} \text{curl}\, F \cdot \vec{n}\, d\sigma = -\int_{\partial S_2} F\cdot d\vec{r}$$



since $\partial S_1 = C$
and $\partial S_2 = -C$

Note that Green's Theorem can be regarded as a special case of Stokes' Theorem by viewing the plane $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$:



$$S \subseteq \mathbb{R}^2 = \{z=0\}$$

$$F(x,y,z) = (P(x,y), Q(x,y), 0)$$

$$\text{curl } F = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{pmatrix}$$

$$= \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Note that $\vec{n} = (0,0,1)$ is the corrected unit normal to S. Hence, by Stokes' Theorem

$$\int_C F \cdot d\vec{r} = \iint_S \text{curl } F \cdot \vec{n} \, d\sigma$$

$$= \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

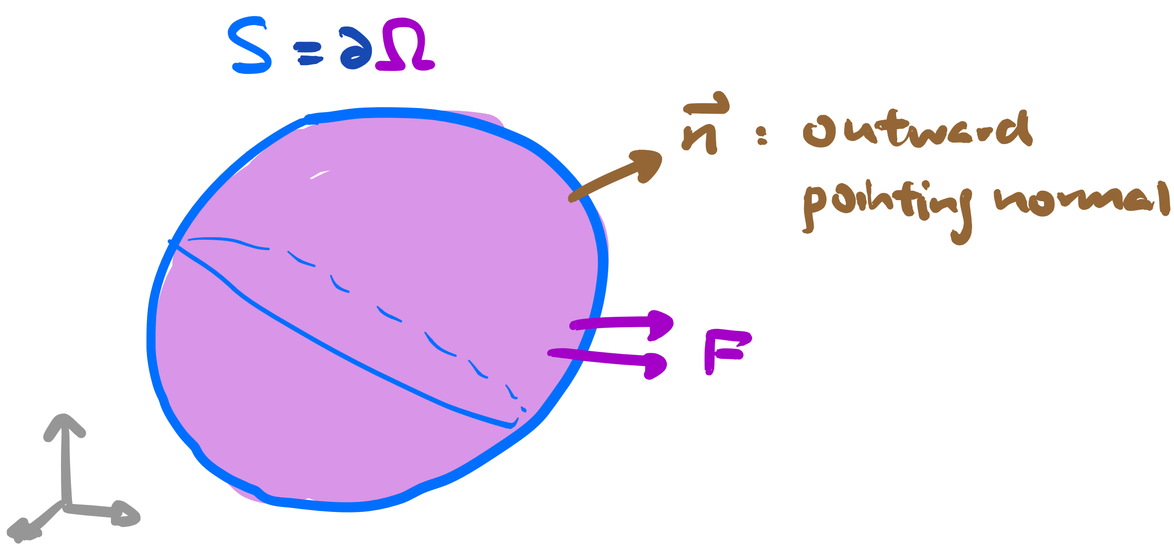
which is exactly Green's Theorem!

Now, we look at another "fundamental theorem" of calculus in $\mathbb{R}^3$.

Divergence Theorem

Let $\Omega \subseteq \mathbb{R}^3$ be an open set bounded by a closed surface $S = \partial\Omega$, oriented by the unit normal $\vec{n}$ pointing out of Ω. THEN: for any C^1 vector field F defined on an open set containing $\overline{\Omega}$, we have

$$\boxed{\iint_S F \cdot \vec{n}\, d\sigma = \iiint_\Omega \operatorname{div} F \cdot dV}$$

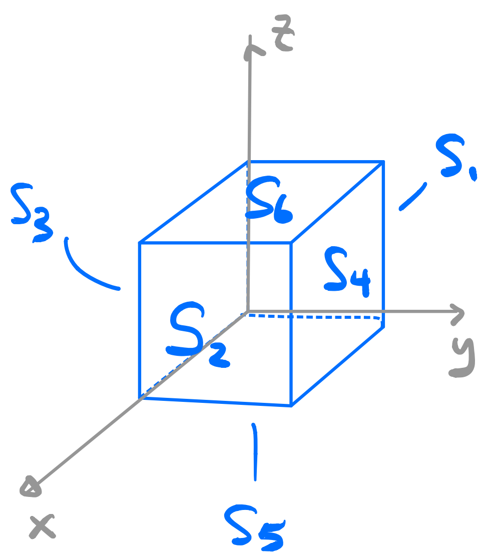


Remark: (1) The theorem still holds when $S = \partial\Omega$ is only piecewise smooth (e.g. Ω = cube)

(2) There is a version of Divergence Theorem in $\mathbb{R}^2$ (in fact, in $\mathbb{R}^n$).

Example 2: Compute the flux $\iint_S F \cdot \vec{n}\, d\sigma$ for the vector field $F(x,y,z) = (x^2, y^2, z^2)$ and the unit cube $S = [0,1] \times [0,1] \times [0,1]$, oriented by the outward pointing normal.

Solution: **Method 1:** Direct computation



Write S as the union of 6 flat pieces and compute the flux over each piece.

$\boxed{\text{On } S_1}$ $x = 0, \quad 0 \le y, z \le 1$

$\vec{n} = (-1, 0, 0)$ pointing out of cube

$F \cdot \vec{n} = -x^2$

Therefore, $\displaystyle\iint_{S_1} F \cdot \vec{n} \, d\sigma = \iint_{S_1} (-x^2) \, d\sigma = 0$

$\boxed{\text{On } S_2}$, $x = 1, \quad 0 \le y, z \le 1$

$\vec{n} = (1, 0, 0)$ pointing out of cube

$F \cdot \vec{n} = x^2$

Therefore, $\displaystyle\iint_{S_2} F \cdot \vec{n} \, d\sigma = \iint_{S_2} x^2 \, d\sigma = 1$

Similarly, we can compute

$$\iint_{S_3} F \cdot \vec{n} \, d\sigma = \iint_{S_5} F \cdot \vec{n} \, d\sigma = 0$$

$$\iint_{S_4} F \cdot \vec{n} \, d\sigma = \iint_{S_6} F \cdot \vec{n} \, d\sigma = 1$$

Hence, the total flux $\displaystyle\iint_{S} F \cdot \vec{n} \, d\sigma = 3$ *

Method 2: Apply Divergence Theorem.

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x+y+z)$$

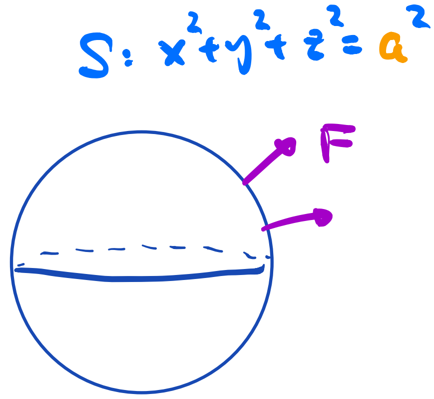
$$\iint_S F \cdot \vec{n}\, d\sigma = \iiint_\Omega \operatorname{div} F\, dV$$

$$= \int_0^1 \int_0^1 \int_0^1 2(x+y+z)\, dx\, dy\, dz$$

$$= 3 \quad \#$$

$\square$

Example 3: Compute the flux of the vector field $F = (x, y, z)$ over the sphere S of radius $a > 0$ centered at the origin, oriented by outward unit normal $\vec{n}$.



Solution: $\vec{n} = \frac{1}{a}(x, y, z)$

$$\iint_S F \cdot \vec{n}\, d\sigma = \iint_S a\, d\sigma = 4\pi a^3 \quad \#$$

$\square$